June 8th, 1876.

Prof. H. J. S. SMITH, F.R.S., President, in the Chair.

Dr. Lindemann and Mr. W. S. W. Vaux were present as visitors.

Mr. Kempe spoke on "A general Method of Describing Plane Curves of the nth degree by Linkwork." Prof. Cayley and Mr. Roberts also spoke on the subject. Mr. Roberts then gave an account of a "Further Note on the Motion of a Plane under certain Conditions." Mr. Walker communicated a short Note "On the reduction of the Equation (U = 0) of the Plane Nodal Cubic to its canonical form $ax^3+by^3+6mxyz$." It was shown that the equation to the three lines of reference is $TU+8SH=0$, and that the equation to (xy) is $L^2-MN=0$, where L, M, N are the factors of $TU-24SH$. The quotient of $TU+8SH$ by L^2-MN will therefore give the third line z, which is the real axis of inflexion, the other two being the nodal tangents.

Prof. Cayley described a surface, depending upon the sinusoid, which was being constructed for him at Cambridge. The Chairman made a few remarks in connection with M. Hermite's recent Note on a Theorem of Eisenstein's.

On a General Method of describing Plane Curves of the n^{th} degree by Linkwork. By A. B. Kempe, B.A.

[*Read June 9th*, 1876.]

Lemma I.—*The Reversor.*—Let $O\xi\beta\alpha$ (Fig. 1) be the linkage known as the contra-parallelogram, $O\xi$ being equal to $\beta\alpha$, and $O\alpha$ to $\xi\beta$.

Make $\alpha\gamma$ a third proportional to $O\xi$ and $O\alpha$, and add the links $O\delta$, equal to $\alpha\gamma$, and $\delta\gamma$, equal to $O\alpha$.

Then the figure $O\alpha\gamma\delta$ is a contra-parallelogram similar to $O\xi\beta\alpha$; and the angle $\xi O\alpha$ is equal to the angle $\delta O\alpha$.

Thus, if $O\xi$ be made to make any angle with $O\alpha$, $O\delta$ will make the same angle with $O\alpha$ on the other side of it.*

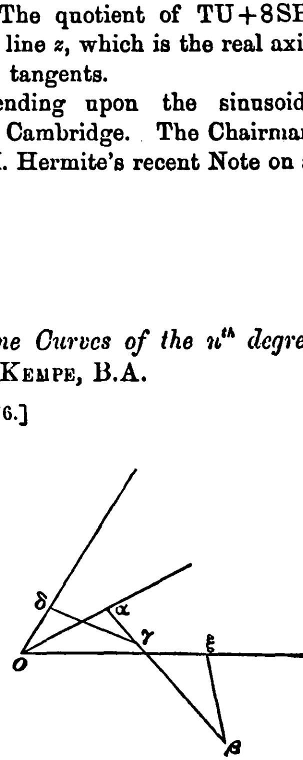

Fig. 1.

* This linkage, and the one next described, were first given by me in the "Messenger of Mathematics," Vol. IV., pp. 122, 123, in a paper "On some new Linkages," §§ 4 and 8.

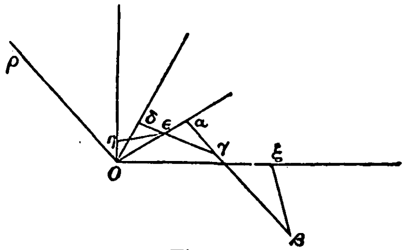

Fig. 2.

Lemma II.—*The Multiplicator.*—In the linkage in Lemma I., if $O\alpha$ make an angle θ with $O\xi$, $O\delta$ will make an angle 2θ.

Now, if (Fig. 2) we add two links $O\eta$, $\eta\epsilon$ to the linkage $O\alpha\gamma\delta$, in the same manner as we added the links $O\delta$, $\delta\gamma$ to the linkage $O\xi\beta\alpha$, we shall get a link $O\eta$ making an angle 3θ with $O\xi$; and, continuing the construction, we shall get a link $O\rho$ making an angle $r\theta$ with $O\xi$, where r is any integer.

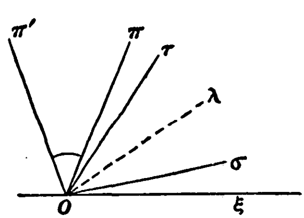

Fig. 3.

Lemma III.—*The Additor.*—By means of the Reversor (Lemma I.) let the links $O\xi$, $O\pi$ (Fig. 3) be made to make equal angles on each side of the link $O\lambda$. Similarly, let the links $O\sigma$, $O\tau$ be made to make equal angles on each side of $O\lambda$.

Then the angle $\pi O\tau$ equals the angle $\sigma O\xi$; thus, if the angle $\sigma O\xi$ equals $s\theta$ and the angle $\tau O\xi$ equals $r\phi$, the angle $\pi O\xi$ equals $r\phi+s\theta$; or if the angle $\sigma O\xi$ equals $s\theta$ and the angle $\pi O\xi$ equals $r\phi$, then the angle $\tau O\xi$ equals $r\phi-s\theta$. Also, if the link $O\pi'$ be fixed to $O\pi$ (or $O\tau$) so as to make an angle α with it, then the angle $\pi' O\xi$ equals $r\phi+s\theta\pm\alpha$ (or $r\phi-s\theta\pm\alpha$).

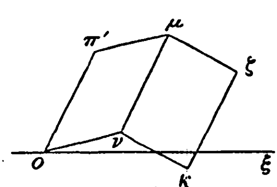

Fig. 4.

Lemma IV.—*The Translator.*—If $O\pi'$ be a link making any angle with $O\xi$ (Fig. 4), and if the two linkage parallelograms $O\pi'\mu\nu$, $\mu\nu\kappa\zeta$ be constructed, it is clear that the link $\kappa\zeta$ is equal and parallel to $O\pi'$, but is otherwise free to assume any position.

Let $\phi(x, y) = 0$ be the equation to any plane curve of the n^{th} degree, and let P (Fig. 5) be any point on the curve.

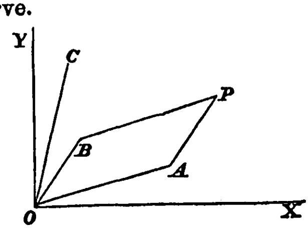

Fig. 5.

Construct the linkage parallelogram OAPB, in which

$$OA = BP = a,$$

$$OB = AP = b;$$

and let the angle $AOX = \theta$ and the angle $BOX = \phi$.

Then $$x = a\cos\theta + b\cos\phi,$$
$$y = a\cos\left(\theta - \frac{\pi}{2}\right) + b\cos\left(\phi - \frac{\pi}{2}\right).$$

Substitute these values of x and y in $\phi(x, y)$, expand and convert powers of cosines into cosines of multiple angles, and then the products of cosines into the cosines of the sums and differences of angles; we shall then get

$$\phi(x, y) = \Sigma\left[A\cos(r\phi \pm s\theta \pm a)\right] + C,$$

where r and s are positive integers, and $a = \frac{\pi}{2}$ or 0, and A and C are constants.

Now, by means of the Multiplicator and Additor, we may make a link OC make an angle COX, with OX equal to $r\phi \pm s\theta \pm a$.

If then OC equals A, the distance of C from OY equals

$$A\cos(r\phi \pm s\theta \pm a).$$

Make then a series of links OC_1, OC_2, &c. make angles with OX equal to the angles of the expression $\Sigma[\quad]$, and let their lengths be the coefficients of the corresponding cosines. Then the sum of the distances of C_1, C_2, &c. from OY will equal

$$\Sigma\left[A\cos(r\phi \pm s\theta \pm a)\right].$$

If then, by means of the Translator, we construct the chain of links OP_1, P_1P_2, &c. (Fig. 6), each link of which is parallel and equal to one of the links OC_1, OC_2, &c.; the point P′, the extremity of the chain, will be at a distance from OY equal to

$$\Sigma\left[A\cos(r\phi \pm s\theta \pm a)\right] = \phi(x, y) - C.$$

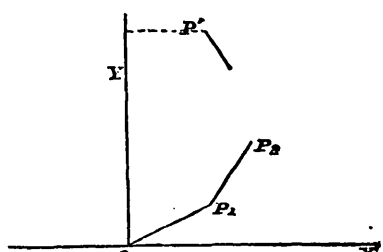

Fig. 6.

But since P lies on the curve, therefore $\phi(x, y) = 0$;

therefore P′ lies at a constant distance $-C$ from OY; *i.e.*, it describes the straight line $$x + C = 0.$$

Conversely, if P′ be made by any of the known "exact parallel motions" to describe the straight line

$$x + C = 0,$$

P will describe the curve $\phi(x, y) = 0$.

It is hardly necessary to add, that this method would not be practically useful on account of the complexity of the linkwork employed, a necessary consequence of the perfect generality of the demonstration.

The method has, however, an interest, as showing that there *is* a way of drawing any given case; and the variety of methods of expressing particular functions that have already been discovered renders it in the highest degree probable that in every case a simpler method can be found. There is still, therefore, a wide field open to the mathematical artist to discover the simplest linkworks that will describe particular curves.

The extension of this demonstration to curves of double curvature and surfaces clearly involves no difficulty.

Further Note on the Motion of a Plane under certain Conditions.

By SAMUEL ROBERTS.

[*Read June 9th*, 1876.]

IN my paper "On the Motion of a Plane under certain Conditions" (Proc., Vol. III., p. 286 *et seq.*), the movement was determined by the paths of two of the points of the plane over a coincident stationary plane. I propose now to consider briefly two other elementary cases to which I referred at the close of my former paper, and relative to which I had then obtained some general results.

Those cases are: (1) when the motion is determined by the path of a point of the moving plane and by the envelope of one of its lines; (2) when the motion is determined by the envelopes of two of the lines of the plane.

(A.) *Line-directrix and Point-directrix.*

(1.) *Locus of Points.*

In the first case the problem is to determine the characteristics of the locus of a point P carried by a constant angle ABC moving in its plane, so that while the side AB always touches a fixed curve, which I call the line-directrix, the other side moves with a fixed point C always on another fixed curve, which I call the point-directrix. The two directrices are supposed to lie in the same plane as the moving angle.

I consider in the first instance a simple case, namely, when the line-directrix is a point, and the point-directrix is a straight line.

The locus is a circular curve of the fourth order and sixth class, having usually two finite double points and another at infinity, where the point-directrix meets the line at infinity. The line-directrix is the focus formed by the circular asymptotes.

If, however, the apex B moves on the point-directrix, then the locus